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STABILITY OF A FLUID-FILLED GYROSCOPE

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An investigation is made of the stability of steady rotation of a symmetrical body with a viscous fluid on the basis of integro-differential equations whose coefficients are determined by solving boundary-value problems of hydromechanics of an ideal fluid that are dependent on the geometry of a cavity. The perturbation method is employed to solve the problem on the stability of body rotation relative to the axis with the largest moment of inertia and on the instability relative to the axis with the smallest moment of inertia. A similar problem for a body with an ideal fluid is studied in [F. L. Chernous'ko, Prikl. Mat. Mekh., 31, Issue 3, 414–432 (1961); S. L. Sobolev, Prikl. Mekh. Tekh. Fiz., No. 3, 3–37 (1960); A. Yu. Ishlinskii and M. E. Temchenko, Prikl. Mekh. Tekh. Fiz., No. 3, 163–179 (1960)], while with a viscous fluid, it is studied in [N. N. Moiseev and V. V. Rumyantsev, Dynamics of a Body with Cavities Filled by Fluid [in Russian], Moscow (1965)], where consideration has been given to the problem on two-dimensional oscillations of a rectangular vessel under the action of the restoring force of an elastic spring.

In describing the perturbed motion of the body with a cavity filled with a viscous fluid, determination is made at first of the velocity field of the ideal fluid filling entirely the cavity of the rigid body. In so doing, a linear problem for eigenvalues is solved. It characterizes natural oscillations of the fluid in a fixed vessel and depends on the geometry of a cavity. Knowing the Zhukovskii potentials and natural oscillations of the fluid, one can determine the coefficients characterizing the inertia coupling of the body and fluid in the cavity as the body moves.

The motion of the entire system is described by an infinite system of integro-differential equations. We will reduce this system in accordance with [6].

Equations of the Perturbed Motion of a Rigid Body with a Fluid. We will consider the perturbed, relative to the steady rotation, motion of a dynamically symmetrical body with an axisymmetrical cavity entirely filled with a low-viscosity incompressible fluid. The angular velocity of the body is presented as $\omega = \omega_0 + \mathbf{\Omega} = \omega_0\mathbf{k} + \mathbf{\Omega}$. Here, $\omega_0 = \omega_0\mathbf{k}$ is the angular velocity of steady rotation of the body directed along the unit vector $\mathbf{k}$ of the Ox_3-axis rigidly connected to the body of the coordinate system $Ox_1x_2x_3$, and $\mathbf{\Omega} = (\Omega_1, \Omega_2, 0)$ is the angular velocity of the body in perturbed motion that represents the value of the first order of smallness as compared to ω_0.

The equations of perturbed motion can be written in the form [6]

$$A\dot{\Omega} + i\,(C - A)\,\omega_0\Omega + 2\rho \sum_{n=1}^{\infty} a_n\,(\dot{s}_n - i\omega_0 s_n) = M\,, \tag{1}$$

$$\mu_n^2 \left(\dot{s}_n - i\lambda_n s_n + \sqrt{\frac{\nu}{\pi}} \int_0^t \frac{\alpha_n\,(t-\tau)\,\dot{s}_n\,(\tau) + \beta_n\,(t-\tau)\,s_n\,(\tau)}{\sqrt{t-\tau}}\,d\tau \right) +$$

$$+ \mathbf{a}_n^+ \dot{\Omega} = -\sqrt{\frac{\nu}{\pi}} \sum_{\substack{m=1 \\ n\neq m}}^{\infty} \int_0^t \frac{\alpha_{mn}\,(t-\tau)\,s_n\,(\tau) + \beta_{mn}\,(t-\tau)\,s_n\,(\tau)}{\sqrt{t-\tau}}\,d\tau\,. \tag{2}$$

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Here, $\Omega = \Omega_1 + i\Omega_2$ and A and C are the moments of inertia of the system body + fluid relative to the axis of symmetry and transverse axis. In this case, Eq. (1) describes the perturbed rotation of a rigid body with a cavity filled with a viscous fluid under action of external moments, while Eq. (2) serves for determination of the amplitudes of fluid oscillations $s_n(t)$.

Let us introduce functions $\varphi_n(x, R)$ satisfying the two-dimensional boundary-value problem:

$$\frac{\partial^2\varphi_n}{\partial R^2} + \frac{1}{R}\frac{\partial\varphi_n}{\partial R} - \frac{\varphi_n}{R^2} + (1-\chi_n^2)\frac{\partial^2\varphi_n}{\partial x^2} = 0\,, \quad \left(\frac{\partial\varphi_n}{\partial R} + \chi_n\frac{\varphi_n}{R}\right)\nu_R + (1-\chi_n^2)\frac{\partial\varphi_n}{\partial x}\,\nu_x = 0\,. \tag{3}$$

The first equation must be satisfied in the meridian section G of the cavity in the plane of cylindrical coordinates (x, R), and the second equation, at the boundary of this section having a normal $\nu = (\nu_R, \nu_x)$; $\chi_n = 2\omega_0/\lambda_n$. Coefficients of the equations are expressed in terms of the functions φ_n as follows:

$$a_n = -\frac{\pi\rho\chi_n}{2\omega_0}\int_G\left[R\frac{\partial\varphi_n}{\partial x} + \frac{\chi_n}{\chi_n - 1}\frac{\partial\varphi_n}{\partial R} + \frac{\varphi_n}{R}\right]R\,ds\,,$$

$$\mu_n = \frac{\pi\rho\chi_n^2}{2\omega_0^2}\int_G\left\{\left(\frac{\partial\varphi_n}{\partial x}\right)^2 + \frac{\chi_n^2+1}{(\chi_n^2-1)^2}\left[\left(\frac{\partial\varphi_n}{\partial R}\right)^2 + \frac{\varphi_n^2}{R^2}\right] + \frac{4\chi_n}{(\chi_n^2-1)^2}\frac{\partial\varphi_n}{\partial R}\frac{\varphi_n}{R}\right\}R\,ds\,, \tag{4}$$

$$\alpha_n(t-\tau) = A_n + B_n\cos 2\omega_0(t-\tau) + 2iC_n\sin 2\omega_0(t-\tau)\,,$$

$$\beta_n(t-\tau) = 2\omega_0 B_n\sin 2\omega_0(t-\tau) - 4i\omega_0 C_n\cos 2\omega_0(t-\tau)\,,$$

where for the cylindrical cavity of radius $r_0 = 1$ and height h in [6] the following coefficients are determined:

$$A_n = T_n h\,(1+k_l^2)(\chi_n^2-1)\,,\ B_n = T_n\,[(1+k_l^2)(\chi_n^2+1) - 2\chi_n]\,,\ C_n = T_n\,(1+k_l^2)(\chi_n^2-1)\,,$$

$$T_n = \pi\rho\chi_n^2/2\omega_0^2(\chi_n^2-1)\,,\quad k_l = \frac{\pi}{2h}(2l+1)\,,\quad l = 0, 1, 2, \ldots\,. \tag{5}$$

By virtue of the fact that the cross coefficients of inertial couplings α_{mn}, β_{mn} exert a weak influence on the dynamics of the rotor with a viscous fluid, we can neglect their influence and leave the main terms with $m = n$. Let us write the system of Eqs. (1) and (2) for a freely rotating body by setting $M = 0$:

$$A\dot\Omega + i(C-A)\,\omega_0\Omega + 2\rho\sum_{n=1}^{\infty} a_n(\dot s_n - i\omega_0 s_n) = 0\,,$$

$$\mu_n\left(\dot s_n - i\lambda_n s_n + \sqrt{\frac{\nu}{\pi}}\int_0^t\frac{\alpha_n(t-\tau)\,\dot s_n(\tau) + \beta_n(t-\tau)\,s_n(\tau)}{\sqrt{t-\tau}}\,d\tau\right) + a_n\dot\Omega = 0\,. \tag{6}$$

A study of the stability of the motion described by Eqs. (6) encounters great difficulties since it necessitates elimination of the parameters s_n from the system of the integro-differential equations. This problem is solved in the next section by the method of perturbation theory.

Perturbation Method. Let us apply the Laplace transformation to Eqs. (6):

$$L(f) = \int_0^{\infty}\exp(-p\tau)\,f(\tau)\,d\tau = \hat f\,.$$

We arrive at

$$[Ap + i\,(C - A)\,\omega_0]\,\hat{\Omega} + 2\rho \sum_{n=1}^{\infty} a_n\,(p - i\omega_0)\,\hat{s}_n = 0\,,$$

$$\mu_n \left[p - i\lambda_n + \frac{1}{2}\sqrt{\nu}\,B_n\,(p + 2\omega_0) \left(\frac{1}{\sqrt{p + i2\omega_0}} + \frac{1}{\sqrt{p - i2\omega_0}} \right) - \right. \tag{7}$$

$$\left. - \sqrt{\nu}\,C_n\,(p - 2\omega_0) \left(\frac{1}{\sqrt{p + i2\omega_0}} - \frac{1}{\sqrt{p - i2\omega_0}} \right) + \frac{A_n}{p}\sqrt{\nu} \right] \hat{s}_n + a_n^* p \hat{\Omega} = 0\,.$$

The characteristic equation of system (7) has the form

$$Ap + i\,(C - A)\,\omega_0 - p\,(p - i\omega_0) \sum_{n=1}^{\infty} \frac{E_n}{\psi_n\,(p)} = 0\,, \tag{8}$$

where

$$E_n = 2\rho a_n^2 / \mu_n\,;$$

$$\psi_n\,(p) = p - i\lambda_n + \frac{A_n}{p}\sqrt{\nu} + \frac{1}{2}\sqrt{\nu}\,B_n\,(p + 2\omega_0) \times$$

$$\times \left(\frac{1}{\sqrt{p + i2\omega_0}} + \frac{1}{\sqrt{p - i2\omega_0}} \right) - \sqrt{\nu}\,C_n\,(p - 2\omega_0) \left(\frac{1}{\sqrt{p + i2\omega_0}} - \frac{1}{\sqrt{p - i2\omega_0}} \right). \tag{9}$$

We will calculate roots of the function $\psi(p)$ using the perturbation method and restricting ourselves to the linear, with respect to a small parameter $\sqrt{\nu}$, terms. We assume that

$$p_n = i\lambda_n + \sqrt{\nu}\,\delta_n\,, \quad \sqrt{\nu} << 1\,, \quad \psi_n\,(p_n) = 0\,. \tag{10}$$

Then

$$\delta_n = \frac{1}{\sqrt{2\omega_0 + \lambda_n}}\,[B_n\,(2\omega_0 - \lambda_n) + C_n\,(2\omega_0 + \lambda_n)] + i \left\{ \frac{A_n}{\lambda_n} + \frac{1}{\sqrt{2\omega_0 + \lambda_n}}\,[C_n\,(2\omega_0 + \lambda_n) - B_n\,(2\omega_0 - \lambda_n)] + \right.$$

$$\left. + \frac{1}{\sqrt{2\omega_0 - \lambda_n}}\,[B_n\,(2\omega_0 + \lambda_n) + C_n\,(2\omega_0 - \lambda_n)] \right\}. \tag{11}$$

Bearing in mind that the loss of stability begins at the frequencies close to the partial frequencies of fluid oscillations, we will expand a meromorphic function $1/\psi_n(p)$ into a Laurent series and restrict ourselves to the terms of the expansion in the neighborhood of the poles, i.e., the zeros of the function $\psi_n(p)$. Then characteristic equation (7) acquires the form

$$Ap + i\,(C - A)\,\omega_0 - p\,(p - i\omega_0) \sum_{n=1}^{\infty} \frac{E_n}{(p - p_n)\,\psi_n'\,(p_n)} = 0\,. \tag{12}$$

Let us calculate a derivative $\psi_n(p_n)$ by using only terms of order $\sqrt{\nu}$:

$$\psi_n'(i\lambda_n) = 1 + \sqrt{\nu}\,\tilde{\alpha}_n + i\sqrt{\nu}\,\tilde{\beta}_n , \tag{13}$$

where

$$\tilde{\alpha}_n = \frac{A_n}{\lambda_n^2} + \frac{B_n}{4\sqrt{2}}\left\{2(\gamma^+ + \gamma^-) - 2\omega_0(\gamma^{+3} + \gamma^{-3}) + \lambda_n(\gamma^{+3} - \gamma^{-3})\right\} - \frac{1}{2\sqrt{2}}C_n\left\{2(\gamma^+ + \gamma^-) + \right.$$

$$\left. + 2\omega_0(\gamma^{+3} - \gamma^{-3}) + \lambda_n(\gamma^{+3} + \gamma^{-3})\right\},$$

$$\tilde{\beta}_n = \frac{1}{4\sqrt{2}}B_n\left\{2(\gamma^- - \gamma^+) - 2\omega_0(\gamma^{+3} - \gamma^{-3}) - \lambda_n(\gamma^{+3} + \gamma^{-3})\right\} - \frac{1}{2\sqrt{2}}C_n\left\{2(\gamma^- - \gamma^+) + \right. \tag{14}$$

$$\left. + 2\omega_0(\gamma^{+3} + \gamma^{-3}) - \lambda_n(\gamma^{+3} - \gamma^{-3})\right\}$$

($\gamma^{\pm} = 1/\sqrt{2\omega_0 \pm \lambda_n}$). Equation (12) with account for (13) and (14) acquires the form

$$Ap + i(C - A)\omega_0 - p(p - i\omega_0)\sum_{n=1}^{\infty}\frac{L_{1n} - iL_{2n}}{p - p_n} = 0 . \tag{15}$$

Here

$$L_{1n} = E_n(1 - \sqrt{\nu}\,\tilde{\alpha}_n) , \quad L_{2n} = \tilde{\beta}_n E_n\sqrt{\nu} . \tag{16}$$

Equation (15) is a generalization of the characteristic equation in [7] to the case of a low viscosity of the fluid.

In the absence of viscosity ($\sqrt{\nu} = 0$) in the case of an ellipsoidal cavity, Eq. (15) coincides with the characteristic equations obtained in [2, 3] provided the gravitational force is equal to zero, while in the case of an arbitrary cavity of rotation it coincides with the equation in [5].

At $\nu = 0$ and with replacement of $p = i\eta$, for the steady rotation to be stable all roots η must be real. As numerical analysis shows [7], in a first approximation instead of the infinite sum only the main term ($\eta = 1$) can be left. The equations for the domains of stability satisfy the following equalities:

$$\Delta = C - A = -E_1 - (A - 2E_1)\frac{2}{\chi_1} \pm \frac{2}{\chi_1}\sqrt{(A - E_1)E_1^2 - (\chi_1 - 2)} , \tag{17}$$

Here, the instability region lies between the curves determined by positive and negative values of the radical. From expression (17) it follows that rotation of the body is stable at $C > A$, i.e., when the body rotates about the axis with the largest moment of inertia.

We will prove that free rotation of the body with an axisymmetrical cavity about the axis of the smallest moment of inertia ($\Delta < 0$) will always be unstable.

Considering in Eq. (15) at $\nu = 0$ the limiting case $\omega_0 \to 0$, we can show that the quantity

$$A' = A - \sum_{n=1}^{\infty} E_n > 0 \tag{18}$$

is equal to the moment of inertia of the equivalent rigid body introduced by Zhukovskii [8].

By virtue of the fact that the infinite sum $\sum_n E_n$ is limited by the value of the moment of inertia A we can suppose that infinite system (15) can be reduced. Let us write characteristic equation (15) by leaving a finite number (N) of terms of an infinite series in it:

$$A\eta + \Delta - \eta\,(\eta - 1)\sum_{n=1}^{N}\frac{E_n}{\eta - \sigma_n} = 0\,, \tag{19}$$

where $\sigma_n = \lambda_n / \omega_0$. In the small neighborhood of the point $\eta = \sigma_n$, Eq. (19) is equivalent to the quadratic equation

$$\eta^2\,(A - E_n) + \eta\,(\Delta + E_n - A\sigma_n) - \Delta\sigma_n = 0\,. \tag{20}$$

The quadratic equation has complex roots, which corresponds to unstable rotation of the body if its determinant is negative:

$$D = (\Delta + E_n - A\sigma_n)^2 + 4\Delta\sigma_n\,(A - E_n) < 0\,. \tag{21}$$

The equality $D = 0$, which represents a quadratic equation for σ_n, gives the intervals of dimensionless natural frequencies corresponding to unstable rotation. These frequencies lie between roots σ_n^0 of the equation $D = 0$:

$$\sigma_n^0 = \frac{1}{A^2}\left[-\Delta A + E_n A + 2\Delta E_n \pm \sqrt{(\Delta A - E_n A - 2\Delta E_n)^2 - A^2\,(\Delta^2 + E_n^2 + 2\Delta E_n)}\,\right]. \tag{22}$$

The expression under the radical is always positive:

$$(\Delta A - E_n A - 2\Delta E_n)^2 - A^2\,(\Delta^2 + E_n^2 + 2\Delta E_n) = 4\Delta C E_n\,(E_n - A) > 0\,,$$

since $\Delta < 0$ and $A > \sum\limits_n E_n > E_n$. At $E_n \to 0$ we have $\sigma_n^0 \to 1 - \dfrac{C}{A} < 1$.

Since natural frequencies possess everywhere a dense spectrum in the region $|\sigma_n| < 1$, we can always choose the actual value of σ_n in the interval determined by formula (22) with the sufficiently, perhaps, small coefficient E_n corresponding to it, in the neighborhood of which Eq. (20) has complex roots at $\Delta < 0$, which is indicative of the instability of rotation of the body about the axis with the smallest moment of inertia. It is easy to evaluate the imaginary part of root (22) determining the order of intensity of the stability loss of steady rotation (i.e., an increment of the rotation axis) at a given frequency σ_n:

$$\mathrm{Jm}\,\eta = \sqrt{-D}\,/A\,.$$

Substituting the mean frequency from interval (22) into expression (21) for D, we arrive at

$$\mathrm{Jm}\,\eta = \sqrt{\left(1 - \frac{E_n}{A}\right)\frac{E_n}{A}\left(1 - \frac{C}{A}\right)\frac{C}{A}}\,.$$

Whence, we obtain the characteristic time of stability loss of the body rotating with the angular velocity ω_0:

$$T = \frac{1}{\omega_0}\left[\left(1 - \frac{E_n}{A}\right)\frac{E_n}{A}\left(1 - \frac{C}{A}\right)\frac{C}{A}\right]^{-1/2}.$$

Criterion of Stability according to the Linear Approximation. In accordance with the foregoing, we leave only one term in the infinite sum from Eq. (15) and perform replacements:

$$p = i\eta\,,\quad p_n = i\lambda_n + \sqrt{\nu}\,\delta_n\,.$$

Let us write the characteristic equation in the form

$$A\eta + (C-A)\,\omega_0 - \eta\,(\eta-\omega_0)\,\frac{L_{11}-iL_{21}}{(\eta-\lambda_1)-\sqrt{\nu}\;i\delta_1} = 0 \tag{23}$$

or in the more convenient form

$$A\eta + (C-A)\,\omega_0 - \frac{\eta\,(\eta-\omega_0)}{\eta-\lambda_1}\,E_1\left\{1+\sqrt{\nu}\left(-\alpha_1 - i\beta_1 + i\,\frac{\delta_1}{\eta-\lambda_1}\right)\right\} = 0\,. \tag{24}$$

Next, we designate

$$\Delta = \left(-\alpha_1 - i\beta_1 + i\,\frac{\delta_1}{\eta-\lambda_1}\right).$$

We will seek a viscosity correction for the root by the perturbation method. Let η^0 be the root of the characteristic equation with the ideal fluid:

$$\eta^0 A\,(\eta^0-\lambda_1) - \eta^0\,(\eta^0-\omega_0)\,E_1 + (C-A)\,\omega_0\,(\eta^0-\lambda_1) = 0\,. \tag{25}$$

The characteristic equation with a viscous fluid has the form

$$A\eta\,(\eta-\lambda_1) + (C-A)\,\omega_0\,(\eta-\lambda_1) - \eta\,(\eta-\omega_0)\,E_1\,[1+\sqrt{\nu}\cdot\Delta] = 0\,. \tag{26}$$

We will seek a correction for the root in the form

$$\eta = \eta^0 + \sqrt{\nu}\cdot\Delta^*\,. \tag{27}$$

Substituting (27) into (26) and leaving the terms that are linear with respect to $\sqrt{\nu}$, we find the correction Δ^*:

$$\Delta^* = \frac{\eta^0\,(\eta^0-\omega_0)\,E_1\cdot\Delta}{A\,(2\eta^0-\lambda_1) - E_1\,(2\eta^0-\omega_0) + (C-A)\,\omega_0}\,. \tag{28}$$

For stability of the rotor with a viscous fluid we find p in explicit form:

$$p = i\eta = i\eta^0 + \sqrt{\nu}\;i\Delta^* = i\eta^0 + \sqrt{\nu}\;\Xi\;i\Delta\,,$$

where

$$\Xi = \frac{\eta^0\,(\eta^0-\omega_0)\,E_1}{A\,(2\eta^0-\lambda_1) - E_1\,(2\eta^0-\omega_0) + (C-A)\,\omega_0}\,. \tag{29}$$

Let us consider separately $i\Delta$:

$$i\Delta = -\,i\alpha_1 + \beta_1 - \frac{\delta_{11}}{\eta^0-\lambda_1} - i\,\frac{\delta_{12}}{\eta^0-\lambda_1}\,.$$

Then for p we have the expression

$$p = i\eta^0 + \sqrt{\nu}\;\Xi\left[\beta_1 - \frac{\delta_{11}}{\eta^0-\lambda_1} - i\left(\alpha_1 + \frac{\delta_{12}}{\eta^0-\lambda_1}\right)\right] = i\left[\eta^0 - \sqrt{\nu}\;\Xi\left(\alpha_1 + \frac{\delta_{12}}{\eta^0-\lambda_1}\right)\right] + \sqrt{\nu}\;\Xi\left(\beta_1 - \frac{\delta_{11}}{\eta^0-\lambda_1}\right).$$

Stability of the steady rotation will be provided if the condition $\mathrm{Re}\, p < 0$ is fulfilled, i.e.,

$$\mathrm{Re}\, p = \sqrt{\nu}\, \Xi \left(\beta_1 - \frac{\delta_{11}}{\eta^0 - \lambda_1} \right) < 0 .$$

The analysis carried out allows us to draw the following conclusions: the presence of viscosity leads, first, to the fact that natural (partial) frequencies are shifted by a value proportional to $\sqrt{\nu}$, i.e., by $\sqrt{\nu}\, \Xi \left(\alpha_1 + \frac{\delta_{12}}{\eta^0 - \lambda_1} \right)$. Second, the presence of viscosity leads to a new criterion of stability unlike the ideal fluid where the criterion of stability is the requirement for the roots of the characteristic equation to be real.

Thus, in some cases the viscosity results in stabilization of the steady rotation (where A is the largest moment of inertia), while in others it results in the loss of stability (where A is the least moment of inertia).

If we consider the Cauchy problem and choose initial conditions close to rotation about the major or minor axes of an inertia ellipsoid, then the motion will consist of the uniform rotation about the axis and small oscillations of this axis.

As for the rate of build-up or attenuation of joint oscillations of the body and fluid, it depends on the mass ratio of the fluid and body, which coincides with the conclusions of [9].

If the fluid density ρ is sufficiently small, the natural oscillations will quickly attenuate and the motion will be merely the forced one.

All the quantities characterizing the motion of the body and fluid will depend on time as $\exp(pt)$, which has been considered in the present work in investigating the stability of steady rotation.

NOTATION

s_n, amplitude of the nth tone of fluid oscillations; ρ, fluid viscosity; ν, kinematic viscosity of the fluid; a_n, μ_n, coefficients of the inertia couplings; λ_n, natural frequencies of the fluid oscillations; α_n, β_n, coefficients of the inertia couplings; α_{mn}, β_{mn}, cross coefficients of the inertial couplings; φ_n and ψ_n, eigenfunctions and eigenvalues of the boundary-value problem; i, complex unity; p, parameter of the Laplace transformation; —, derivative with respect to t; M, moment of external forces; σ_n, correction for frequency; $\tilde{\alpha}_n$, $\tilde{\beta}_n$, coefficients of expansion of the function; $\mathrm{Jm}\,\eta$, increment of the axis of rotation.

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